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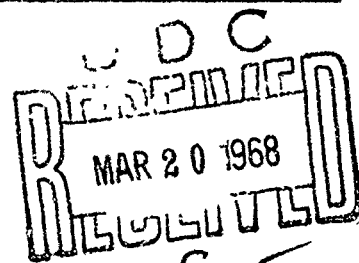
THE ESTIMATION OF PERIODIC SIGNALS IN NOISE

T. G. Kincaid and H. J. Scudder, III

Contract No. NOnr-4692(60)

February 1968

Office of Naval Research



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## ABSTRACT

This report presents two methods for the estimation of a periodic signal in additive noise. Both methods assume that only a finite time sample of the signal plus noise is available for processing. The estimate of the signal is chosen to minimize the mean square error between the estimate and the sample of signal plus noise. This is also a maximum likelihood estimate if the noise is white and Gaussian.

The first method is frequency domain analysis. Estimates for the period of the signal and the complex amplitudes of its harmonics are derived.

The second method is time domain analysis. Estimates for the period of the signal and for the waveform of one period are derived.

Under the assumption of white noise and large signal-to-noise ratio, formulas for the expected values and variances of the period estimates are derived. The estimates for the period are found to be the same by both methods. The estimate is unbiased and has a variance inversely proportional to the signal-to-noise ratio, and inversely proportional to the cube of the number of periods in the given sample. The expected values of the estimates of the waveform itself are derived, and the estimates are found to be biased.

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# THE ESTIMATION OF PERIODIC SIGNALS IN NOISE

T.G. Kincaid\* and H.J. Scudder, III

## I. INTRODUCTION

In this report we consider the problem of estimating a periodic signal in additive noise. Although this is a special application of the established theory of parameter estimation, (1, 2) periodic signals deserve a detailed treatment because of their frequent occurrence. For example, the results of this study are useful in the analysis of sounds produced by rotating machinery. In this application, good signal estimates can be used for source identification or malfunction diagnosis.

We shall assume that a finite time sample of the signal plus noise is available for inspection. We shall show two methods of forming the estimate of the periodic signal which minimizes the mean square error between this estimate and the given time sample of signal plus noise. (It is well known that the minimum mean square error estimator is also a maximum likelihood estimator if the noise is white and Gaussian.) The first method uses a frequency domain approach, while the second method works in the time domain.

In the frequency domain method, the first step is to estimate the period by summing the power spectrum of the data at harmonics of a sequence of trial fundamental frequencies. The period of the fundamental which gives the largest sum is the estimate. The Fourier coefficients of desired waveform are then estimated from the values of the Fourier transform of the data at harmonics of the estimated fundamental.

In the time domain method, the period and the waveform are estimated simultaneously. The technique is simply to choose a trial period  $P$ , divide the data into  $P$  length sections, and average the sections. The trial period which produces the average waveform with the largest energy is the estimate of the time period, and the average waveform is the estimate of the true waveform.

Under the assumption of white noise and large signal-to-noise ratio, formulas for appropriate statistics of the estimates are derived for both methods. The mean and variance of the period estimates are found to be the same by both methods. The mean is found to be unbiased. The variance is found to be inversely proportional to the signal-to-noise ratio and inversely proportional to the cube of the number of periods in the data sample. Formulas for the mean of the Fourier coefficients are derived for the frequency domain estimates, and it is found that these estimates are biased. In the time domain approach, it is necessary to assume a noise distribution to compute the mean of the waveform estimate. On the assumption of Gaussian noise, this estimate is shown to be

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biased.

In this report the following notation will be used.

$\tilde{f}(t)$  denotes a function defined on the interval  $-\infty < t < \infty$ , and is a theoretical ideal,

$f(t)$  denotes a function which is equal to  $\tilde{f}(t)$  on a finite interval  $T$ , and zero elsewhere, and is the real signal we must work with.

$$\text{rect}(x) = \begin{cases} 0 & \text{for } |x| > 1/2 \\ 1/2 & \text{for } |x| = 1/2 \\ 1 & \text{for } |x| < 1/2 \end{cases}$$

$$\text{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

(rect and sinc are a Fourier transform pair);

$\delta(t)$  denotes the Dirac delta function;

$$g_x(t) = \begin{cases} 0 & \text{for } t < 0 \text{ and } t \geq x \\ 1 & \text{for } 0 \leq t < x \end{cases}, \text{ the gate function;}$$

\* between functions denotes convolution;

\* as a superscript denotes complex conjugation;

E is an expectation (ensemble averaging) operator.

In general, where small letters are used for time functions, the corresponding capitals are used for their Fourier transforms.

Formally, we initiate our analysis with the following assumptions:

- (i)  $\tilde{z}(t) = \tilde{x}_0(t) + \tilde{n}(t)$  has bandwidth B Hz;
- (ii)  $\tilde{x}_0(t)$  is periodic with period  $P_0 \gg \frac{1}{B}$ , i. e., many harmonics;
- (iii)  $\tilde{n}(t)$  is a sample function of a zero mean wide sense stationary random process with covariance function  $R(t-s)$  and Power Spectrum  $P(f)$ ;
- (iv) For band-limited white noise,  $R(t-s) = \sigma^2 \text{sinc}(2Bt)$  and  $P(f) = \frac{\sigma^2}{2B} \text{rect}\left(\frac{f}{2B}\right)$ ;
- (v)  $\tilde{x}_0(t) = \sum_n w_0(t-nP_0)$  (see Ref. 3)

where  $w_0(t) = 0$ ; for  $t < 0$ ,  $t \geq P_0$ .

$$(vi) \frac{1}{P_0} \int_0^{P_0} w_0^2(t) dt \gg \sigma^2$$

i. e., large signal-to-noise ratio.

$$(vii) \tilde{z}(t) \text{ is known on an interval of length } T \gg P_0$$

i. e., many periods.

## II. THE FREQUENCY DOMAIN APPROACH

In the frequency domain approach we begin by characterizing the unknown periodic waveform  $\tilde{x}_0(t)$  as a Fourier series. The optimization procedure then yields estimators of the period  $P_0$  and Fourier coefficients  $c_{0k}$  of  $\tilde{x}_0(t)$ .

### A. The Estimates of the Period and Fourier Coefficients

Any periodic signal  $\tilde{x}(t)$  can be written as a Fourier series as follows

$$\tilde{x}(t) = \sum_k c_k e^{j2\pi kt/P} \quad (1)$$

As our estimate of  $\tilde{x}_0(t)$ , we seek the particular periodic signal  $\tilde{x}(t) = \hat{x}(t)$  which minimizes the mean square error between  $\tilde{x}(t)$  and the observed data  $\tilde{z}(t)$  on the interval  $-T/2 \leq t \leq T/2$ . That error is given by

$$L = \frac{1}{T} \int_{-T/2}^{T/2} [\tilde{z}(t) - \tilde{x}(t)]^2 dt \quad (2)$$

In Appendix A it is shown that the estimates of the period  $\hat{P}$  and the Fourier coefficients  $\hat{c}_k$  of  $\hat{x}(t)$  are determined as follows.

First compute  $Z(f)$ , the Fourier transform of  $z(t)$ . Then  $\hat{P}$  is the value of  $P$  which maximizes the expression

$$V(P) = \frac{1}{T} \sum_{k=1}^K |Z(k/P)|^2 \quad (3)$$

where  $K$  is the total number of harmonics that can be accommodated by the bandwidth of  $z(t)$ . The Fourier coefficient estimates  $\hat{c}_k$  are the values of  $c_k$  given by

$$\hat{c}_k = \hat{a}_k + j \hat{b}_k \quad (4)$$

where

$$\hat{a}_k = \frac{1}{T} Z_r(k/\hat{P}) \quad (5)$$

$$\hat{b}_k = \frac{1}{T} Z_i(k/\hat{P}) \quad (6)$$

$Z_r(f)$  and  $Z_i(f)$  are, respectively, the real and imaginary parts of  $Z(f)$ . The estimate  $\hat{x}(t)$  of the unknown waveform can then be generated by the relation

$$\hat{x}(t) = \sum_k \hat{c}_k e^{j2\pi kt/\hat{P}} \quad (7)$$

These estimates are independent of the noise covariance  $R(t-s)$ .

The estimates are intuitively what we would expect. The period is estimated by summing the values of the power spectrum of  $z(t)$  at the harmonics of the fundamental frequency  $1/P$ . When plotted as a function of  $P$ , one would expect this sum to be greatest when  $P$  is close to the true period. The Fourier coefficients of  $x_0(t)$  are then just  $1/T$  the values of the Fourier transform at harmonics of the estimated fundamental frequency.

## B. The Resolution of the Period Estimate

In practice,  $V(P)$  cannot be tested for a maximum at every value of  $P$ . Usually  $P_0$  is known approximately (from auto-correlation or power spectrum data), so that a range of values of  $P$  to be tested can be established. In order to reduce the computing time, it is desirable to test as few values of  $P$  as necessary within that range. How far apart can we choose the test values of  $P$  without danger of missing the maximum of  $V(P)$ ? To answer this question we need to know the behavior of  $V(P)$  in the vicinity of the maximum.

Since the Fourier transform is a linear operation, the assumption of a large signal-to-noise ratio carries over into the frequency domain. Thus the gross behavior of  $V(P)$  in the vicinity of the maximum will be dominated by the signal, and we can study that behavior adequately by considering the ideal case in which there is no noise. In Appendix B the noise free behavior of  $V(P)$  is studied in a region  $\Delta P$  about  $P_0$ . For  $h \ll P_0$  it is shown that

$$V(P_0 + h) = \sum_{k=1}^K |c_{0k}|^2 \text{sinc}^2 \left( \frac{kT}{P_0^2} h \right) \quad (8)$$

Equation (8) shows that in the vicinity of  $P_0$  the noise-free  $V(P)$  is a weighted sum of squared sinc functions, centered on  $P_0$ . The widths of the main lobes of the sinc functions are inversely proportional to  $k$ , each lobe going to zero at  $h = P_0^2/kT$ . The "roll off" of  $V(P)$  in the immediate vicinity of  $P_0$  is controlled by the narrowest of these squared sinc functions. This is the term in Eq. (8) for which  $k=K$ . Thus, in order not to miss the maximum,  $V(P)$  should be



sampled at intervals not larger than

$$\Delta P = \frac{P_0^2}{KT} \quad (9)$$

We shall call the quantity  $\Delta P$  given by Eq. (9) the resolution of the period estimator.

Since  $K$  is the total number of harmonics in  $\tilde{x}_0(t)$ , it is appropriate to define the bandwidth  $B_0$  of  $\tilde{x}_0(t)$  to be  $K/P_0$ . Then Eq. (9) can be written

$$\Delta P = \frac{P_0}{B_0 T} \quad (10)$$

Equation (10) is satisfying in its simplicity. It says that the resolution of the period estimator is equal to the true period divided by an appropriately defined time bandwidth product. The resolution does not depend upon the detailed structure of the periodic waveform.

### C. The Expectation and Variance of the Period Estimate

Because of the noise, we cannot expect the maximum of  $V(P)$  to lie exactly at  $P_0$ . As a measure of how close  $\hat{P}$  is likely to come to  $P_0$ , we shall compute the expectation and the variance of the random variable  $\hat{P}$ . We proceed as follows. (4)

$V(P)$  is a well-behaved function of  $P$ . Thus we can expand  $V(P)$  in a Taylor series about the true period  $P_0$ .

$$V(P) = V(P_0 + h) = V(P_0) + V'(P_0) h + V''(P_0) \frac{h^2}{2} + \dots \quad (11)$$

We shall only investigate the region  $h \ll P_0$ , so we can neglect terms of higher than second order in  $h$ . The maximum of  $V(P)$  may be found by taking the derivative of  $V(P)$  and equating it to zero.

$$V'(P) = 0 = V'(P_0) + V''(P_0) h \quad (12)$$

Solving Eq. (12) for  $h$ , we find that to a first approximation

$$\hat{P} = P_0 - \frac{V'(P_0)}{V''(P_0)} \quad (13)$$

In Appendix C it is shown that the expectation of  $\hat{P}$  as given by Eq. (13) is

$$E\hat{P} = P_0 \quad (14)$$

i. e., the estimate is unbiased.

It is also shown in Appendix C that the variance of  $\hat{P}$  as given by Eq. (13) is

$$\text{Var } \hat{P} = \sigma_P^2 = \frac{3 P_0^3}{2 \pi^2 T^3} \frac{1}{B B_w^2} \frac{\sigma^2}{E_w} \quad (15)$$

where  $B_w^2$  is the second moment of the spectrum of  $w_0(t)$ , defined by Eq. (C25), and  $E_w$  is the energy of  $w_0(t)$ , defined by Eq. (C26). Equation (15) shows that the variance of  $P$  decreases as the inverse cube of the length of the data sample  $T$ , and that it decreases as the inverse of the signal-to-noise ratio  $E_w/\sigma^2$ .

#### D. The Expectations of the Fourier Coefficient Estimates

The estimates of the complex Fourier coefficients of the signal are, by Eqs. (4), (5), and (6)

$$\begin{aligned} \hat{c}_k &= \frac{1}{T} Z \left( \frac{k}{P} \right) \\ &= \frac{1}{T} \left[ X_0 \left( \frac{k}{P_0} \right) + \frac{k^2}{P_0^4} X_0'' \left( \frac{k}{P_0} \right) \frac{(\hat{P} - P_0)^2}{2} + \dots + N \left( \frac{k}{P} \right) \right] \end{aligned}$$

Thus

$$\begin{aligned} E \hat{c}_k &= \frac{1}{T} \left[ X_0 \left( \frac{k}{P_0} \right) - \frac{k^2 2 \pi^2 T^2}{P_0^4 3} X_0 \left( \frac{k}{P_0} \right) E \frac{(\hat{P} - P_0)^2}{2} + \dots + 0 \right] \\ &= \frac{1}{T} X_0 \left( \frac{k}{P_0} \right) \left[ 1 - \frac{k^2}{P_0^4} \frac{\pi^2 T^2}{3} \text{Var } \hat{P} \right] \\ &= \frac{1}{T} X \left( \frac{k}{P_0} \right) \left[ 1 - \frac{k^2}{P_0 T} \frac{1}{B B_w^2} \frac{\sigma^2}{E_w} \right] \end{aligned} \quad (16)$$

We see that the estimates of the Fourier coefficients are biased, but that the bias approaches 0 as  $T \rightarrow \infty$ , and decreases as the signal-to-noise ratio increases.

### III. THE TIME DOMAIN APPROACH

We shall assume in this analysis that  $\tilde{z}(t)$ , the periodic signal plus noise, is known on an interval  $0 \leq t < T$ . Let  $g_T(t)$  be the gate function defined in the Introduction. Then

$$z(t) = \tilde{z}(t) g_T(t)$$

is the known signal, and

$$x_0(t) = \tilde{x}_0(t) g_T(t)$$

is the true periodic waveform obscured by noise

$$n_0(t) = \tilde{n}_0(t) g_T(t)$$

on this interval. In this approach we shall find an estimate  $\hat{w}(t)$  of  $w_0(t)$ , the waveform of one period of  $\tilde{x}_0(t)$ ; and we shall find an estimate  $\hat{P}$  of  $P_0$ , the period of  $\tilde{x}_0(t)$ . These two estimates give an estimate  $\hat{x}(t)$  of  $x_0(t)$ . We shall choose as our estimate  $\hat{x}(t)$  the particular  $x(t)$  which minimizes the mean square error

$$L = \frac{1}{T} \int_0^T [z(t) - x(t)]^2 dt \quad . \quad (17)$$

We shall also determine the mean and variance of  $\hat{P}$ , and the mean of  $\hat{w}(t)$  under the assumption of Gaussian noise.

#### A. The Estimate of the Waveform for a Fixed Period

In this section we assume a fixed period  $P$  and find a periodic waveform which minimizes  $L$ . To simplify the analysis we shall minimize  $L$  over a section of data  $NP$  long, rather than  $T$ . The integer  $N$  is chosen so that the interval  $NP$  is as large as possible i. e.,  $NP \leq T < (N+1)P$ . Thus we shall actually find the periodic waveform  $\hat{x}_P(t)$  which minimizes

$$L_P = \frac{1}{NP} \int_0^{NP} [z(t) - x(t)]^2 dt \quad . \quad (18)$$

Let  $\hat{w}_P(t)$  be one period of  $\hat{x}_P(t)$ . In Appendix D it is shown that

$$\hat{w}_P(t) = \frac{1}{N} \sum_{n=0}^{N-1} z(t + nP) g_P(t); \quad NP \leq T < (N+1)P \quad . \quad (19)$$

This equation says that  $\hat{w}_P(t)$  is the average of  $N$  successive  $P$  length sections of  $z(t)$ . This result is intuitively what one would expect. As successive  $P$  length intervals are averaged the periodic component remains constant while the noise tends to average to zero.

It is of interest to examine the error in the estimate  $\hat{w}_P(t)$  caused by restricting the data to a duration  $NP$ . It is shown in Appendix D that the difference between  $\hat{w}_P(t)$  given by Eq. (19) and the estimate  $\hat{w}'_P(t)$  which minimizes  $L$  given by Eq. (17) is

$$\hat{w}'_P(t) - \hat{w}_P(t) = \begin{cases} \frac{1}{N+1} [z(t + NP) - \hat{w}_P(t)]; & 0 \leq t < (T - NP) \\ 0 & ; (T - NP) \leq t < T \end{cases} \quad . \quad (20)$$

This error is the difference between averaging the left over "tail" of the data into the estimate. Since the amplitude of error is inversely proportional to  $N+1$ , our introductory assumption that  $T \gg P$  makes this error small. By neglecting this error we not only simplify the analysis but eliminate an untidy discontinuity in  $\hat{w}_P(t)$  at  $t=T-NP$ .

## B. The Estimate of the Period

Now that we have learned how to form  $\hat{w}_P(t)$  for an arbitrary  $P$ , we shall find an estimate  $\hat{P}$  for the true period  $P_0$ . By our stated criterion, we require  $\hat{P}$  to be that  $P$  which minimizes  $L_P$  as given by Eq. (18). It appears impossible to get any satisfaction by setting the derivative of  $L_P$  with respect to  $P$  equal to zero. The  $P$  is tied up inside the  $x(t)$  in an unknown manner and cannot be brought out. The best technique appears to be brute force computation of  $L$  as a function of  $P$ , and to choose the  $P$  which minimizes  $L$  as  $P$ .

However, it is not necessary to actually go through the computation of  $L_P$  for each  $P$  in order to minimize  $L_P$ . It is sufficient to compute the energy of the estimate  $\hat{w}_P(t)$  for each  $P$ . Then the  $P$  for which this energy is a maximum is the estimate  $\hat{P}$  of the true period  $P_0$ .

To show this, we first define the error signal estimate

$$\hat{e}_P(t) = z(t) - \hat{x}_P(t) \quad . \quad (21)$$

In Appendix E it is shown that  $\hat{e}_P(t)$  is orthogonal to  $\hat{x}_P(t)$ , viz.,

$$\int_0^{NP} \hat{x}_P(t) \hat{e}_P(t) dt = 0 \quad . \quad (22)$$

Now, our objective was to choose  $P$  to minimize the mean square error signal,  $L_P$ .

$$\begin{aligned} L_P &= \frac{1}{NP} \int_0^{NP} [z(t) - \hat{x}_P(t)]^2 dt \\ &= \frac{1}{NP} \left\{ \int_0^{NP} z^2(t) dt - 2 \int_0^{NP} [\hat{x}_P(t) + \hat{e}_P(t)] \hat{x}_P(t) dt + \int_0^{NP} \hat{x}_P^2(t) dt \right\} \\ &= \frac{1}{NP} \left\{ \int_0^{NP} z^2(t) dt - \int_0^{NP} \hat{x}_P^2(t) dt \right\} \quad . \quad (23) \end{aligned}$$

Since the first term is very nearly a constant for a small range of  $P$ ,  $L_P$  will be a minimum when the second term is a maximum. (5) Thus  $\hat{P}$  is the value of  $P$  which maximizes the expression

$$V(P) = \int_0^{NP} \hat{x}_P^2(t) dt = P \int_0^P \hat{w}_P^2(t) dt \quad (24)$$

The choice of notation is deliberate. In Appendix F it is shown that Eqs. (24) and (3) are equivalent definitions of  $V(P)$ .

Since the estimator of the period is the same for both the time and frequency domain approaches, the resolution of the period estimator is given by Eq. (9) or Eq. (10). Similarly, the expectation and variance of the estimate are given by Eqs. (14) and (15), respectively.

### C. The Expected Value of the Waveform Estimate

In this section we shall find the expected value of  $\hat{w}(t)$ , the estimate of the waveform of one period. Unfortunately, it does not seem possible to do this without assuming a probability distribution for the noise. Under the assumption of a Gaussian distribution, it is shown in Appendix G that

$$E w(t) = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} x(t + n P_0) s(t); & t \geq 0 \\ 0 & ; t < 0 \end{cases} \quad (25)$$

where

$$s(t) = 1 - \operatorname{erf} \left( \frac{t - P_0}{\sigma_P} \right)$$

and  $\operatorname{erf}(x)$  is the error function.  $\sigma_P^2$  is the variance of  $\hat{P}$ . The function  $s(t)$  is plotted in Fig. 1. The function  $s(t)$  is a sort of "soft gate," which approaches the gate function  $g_{P_0}(t)$  as the noise variance  $\sigma^2$  approaches zero. This "softness" of the gate reflects the uncertainty in the period of  $\hat{x}(t)$ .

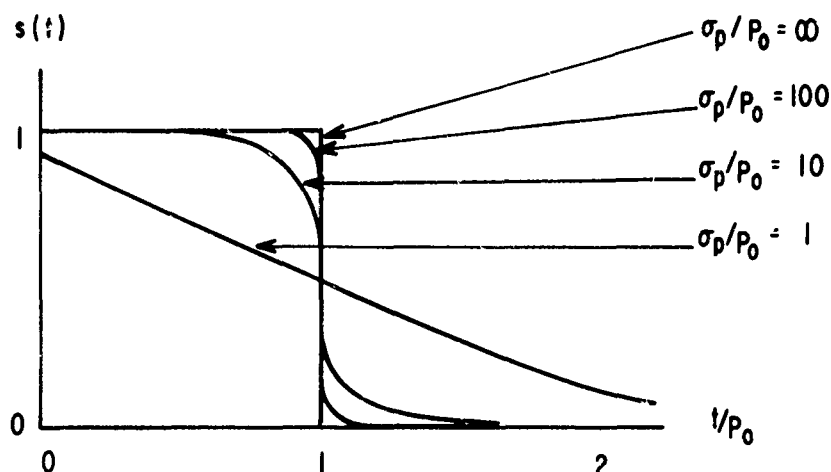


Fig. 1 The expectation of the gate function for various ratios of period variance to true period.

#### IV. CONCLUSION

Two methods of finding the least mean square estimate of a periodic function in noise have been derived. These have been labeled the frequency domain and time domain methods. Under appropriate assumptions about the noise, formulas for the expectation and variance of the period estimate, and the expectation of the waveform estimates have been found. The variances of the waveform estimates were not found.

Of the two methods, the authors have found the time domain method more satisfactory if it is not necessary to compute the Fourier transform for other reasons. If  $n$  data waveform samples are available, and it is desired to search  $m$  periods for a maximum, then the time required is proportional to  $mn$ . The time domain method has the additional feature that the estimates can be made in one pass through the data. This is important if the data are stored on tape, or some other slow memory.

When the Fourier transform of the data has to be computed anyway for reasons other than waveform estimation, the frequency domain method is advantageous. Taking the Fourier transform of  $n$  data points requires a time proportional to  $n \log n$ , assuming the fast Fourier transform is used. Then, once the power spectrum is formed, it can be searched exhaustively for evidence of periodicities. This search requires  $mh$  operations, where  $h$  is the average number of harmonics summed on each period tested. This is essentially a two-pass system. On the first pass the data are Fourier transformed, and on the second pass the power spectrum is searched.

It is worth noting that the period estimator also serves as a suboptimal detector of periodic waveforms. When there are a large number of strong harmonics, it is somewhat more sensitive than a spectrogram, which does not sum the harmonics. The optimum detector takes advantage of the waveform estimate in an estimator-correlator configuration. This will be shown by the authors in a forthcoming report.

## APPENDIX A

We wish to verify that  $\hat{P}$  is the value of  $P$  which maximizes  $V(P)$  given by Eq. (3), and that  $\hat{c}_k$  is the value of  $c_k$  given by Eq. (4) of the main text.

From Eqs. (1) and (2) we have that

$$L = \frac{1}{T} \int_{-T/2}^{T/2} \left[ z^2(t) - 2z(t) \sum_k c_k e^{+j2\pi kt/P} + \sum_k \sum_l c_k c_l e^{j2\pi(k+l)t/P} \right] dt \quad (A1)$$

Thus,

$$L = \frac{1}{T} \int_{-T/2}^{T/2} z^2(t) dt + \sum_k \sum_l c_k c_l \operatorname{sinc} \left[ (k+l) \frac{T}{P} \right] - \frac{2}{T} \sum_k c_k \int_{-T/2}^{T/2} z(t) e^{j2\pi kt/P} dt \quad (A2)$$

If we define

$$Z(k/P) = \int_{-T/2}^{T/2} z(t) e^{-j2\pi kt/P} dt, \quad (A3)$$

the Fourier transform of the input  $z(t)$  at frequency  $k/P$ , and we use the assumption  $T \gg P$ , then

$$L \cong \frac{1}{T} \int_{-T/2}^{T/2} z^2(t) dt + \sum_k |c_k|^2 - \frac{2}{T} \sum_k c_k Z^*(k/P) \quad (A4)$$

since only the diagonal terms of the double sum make major contributions. Let  $c_k = a_k + jb_k$ , and note that  $b_k = b_{-k}$ ,  $a_k = a_{-k}$ .

Then

$$L \cong \frac{1}{T} \int_{-T/2}^{T/2} z^2(t) dt - \sum_k \left( a_k^2 + b_k^2 \right) - \frac{2}{T} \sum_k [a_k Z_r(k/P) - b_k Z_i(k/P)] \quad (A5)$$

Minimizing with respect to the  $a_k$  and  $b_k$  yields

$$a_k = \frac{Z_r}{T}(k/P) \quad b_k = \frac{Z_i}{T}(k/P) \quad (A5)$$

Substituting, we find

$$L \cong \frac{1}{T} \int_{-T/2}^{T/2} z^2(t) dt + \frac{1}{T^2} \sum_k |Z(k/P)|^2 - \frac{2}{T^2} \sum_k |Z(k/P)|^2$$

or

$$L \cong \frac{1}{T} \int_{-T/2}^{T/2} z^2(t) dt - \frac{1}{T^2} \sum_k |Z(k/P)|^2 \quad (A6)$$

Since the first term on the right of Eq. (A6) is a constant,  $L$  is minimized when the second term on the right is maximized. Since  $z(t)$  has some finite practical bandwidth, it is not necessary to sum over all  $k$ , but rather to some upper limit  $K$ . Thus the value of  $P$  which maximizes  $V(P)$  is given by Eq. (3). Equations (A5) give the values of  $a_k$  and  $b_k$  which minimize  $L$  for any choice of  $P$ . Thus  $L$  is minimized with respect to both  $P$  and  $c_k$  when  $c_k = \hat{c}_k$ , as given by Eq. (4).



## APPENDIX B

We wish to verify that the behavior of  $V(P)$  in the vicinity of the maximum is given by Eq. (8) in the main text.

Since we are assuming the ideal case of no noise,  $z(t) = x_0(t)$ . The signal is available between  $-T/2$  and  $T/2$ , so

$$x_0(t) = \text{rect} \left( \frac{t}{T} \right) \sum_n c_{0n} e^{j2\pi n t / P_0} \quad . \quad (B1)$$

The Fourier transform of  $z(t)$  is

$$Z(f) = T \sum_n c_{0n} \text{sinc} T \left( f - \frac{n}{P_0} \right) \quad . \quad (B2)$$

Substituting Eq. (B2) into Eq. (3) for  $V(P)$  gives

$$V(P) = \sum_{k=1}^K \sum_m \sum_n c_{0m} c_{0n}^* \text{sinc} T \left( \frac{k}{P} - \frac{m}{P_0} \right) \text{sinc} T \left( \frac{k}{P} - \frac{n}{P_0} \right) \quad . \quad (B3)$$

If we set  $P = P_0 + h$  and just consider the region where  $h \ll P_0$  then Eq. (B3) can be written

$$V(P_0 + h) = \sum_{k=1}^K \sum_m \sum_n c_{0m} c_{0n}^* \text{sinc} \frac{T}{P_0} \left( k - m - \frac{kh}{P_0} \right) \text{sinc} \frac{T}{P_0} \left( k - n - \frac{kh}{P_0} \right) \quad . \quad (B4)$$

For  $T/P_0 \gg 1$  only the terms for which  $m = k$  and  $n = k$  will contribute significantly to the sum on  $k$ . Setting the terms for which  $m, n \neq k$  equal to zero, Eq. (B4) becomes Eq. (8).

### APPENDIX C

We wish to verify that the expectation of  $P$  is given by Eq. (14), and that the variance is given by Eq. (15). Since the expression for  $V(P)$  involves the Fourier transform  $Z(f)$  of  $z(t)$ , it is a necessary preliminary to investigate the  $Z(f)$ .

From assumption (i) of the Introduction,

$$Z(f) = X_0(f) + N(f) \quad (C1)$$

where  $X_0(f)$  and  $N(f)$  are the Fourier transforms of  $x_0(t)$  and  $n(t)$ , respectively. For the transform of the signal alone we have

$$\begin{aligned} X_0(f) &= \int_{-\infty}^{\infty} \tilde{x}_0(t) \text{rect}(t/T) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} [w_0(t) * \sum_n \delta(t-nP)] \text{rect}(t/T) e^{-j2\pi ft} dt \\ &= \left[ W_0(f) \cdot \frac{1}{P} \sum_n \delta\left(f - \frac{n}{P}\right) \right] * T \text{sinc} T f \\ &= \frac{T}{P} \sum_n W_0\left(\frac{n}{P}\right) \text{sinc} T \left(f - \frac{n}{P}\right) \end{aligned} \quad (C2)$$

We now consider the behavior of the noise process  $N(f)$ , which is a complex random variable for each  $f$ . Its mean is

$$E N(f) = E \int_{-\infty}^{\infty} n(t) e^{-j2\pi ft} dt = \int_{-T/2}^{T/2} E \tilde{n}(t) e^{-j2\pi ft} dt = 0 \quad (C3)$$

The average noise energy density is given by  $E|N(f)|^2$ , and is determined as follows:

$$\begin{aligned} E|N(f)|^2 &= E \left[ \int_{-\infty}^{\infty} \tilde{n}(t) \text{rect}(t/T) e^{-j2\pi ft} dt \right]^2 \\ &= E \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{n}(t) \tilde{n}(s) \text{rect}(t/T) \text{rect}(s/T) e^{-j2\pi f(t-s)} dt ds \\ &= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} R(t-s) e^{-j2\pi f(t-s)} dt ds \end{aligned} \quad (C4)$$

For band-limited white noise

$$\begin{aligned}
 E|N(f)|^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{rect}(t/T) \text{rect}(s/T) \sigma^2 \text{sinc}[2B(t-s)] e^{-j2\pi f(t-s)} dt ds \\
 &= \sigma^2 \int_{-\infty}^{\infty} ds \text{rect}(s/T) e^{j2\pi fs} [\text{sinc}(2Bs) * \text{rect}(s/T) e^{+j2\pi fs}] \quad (C5)
 \end{aligned}$$

Taking the Fourier transform of the convolution, and integrating over  $s$  first, we get

$$E|N(f)|^2 = \frac{\sigma^2 T^2}{2B} \int_{-\infty}^{\infty} \text{sinc}^2 T(f-g) \text{rect}\left(\frac{g}{2B}\right) dg \quad (C6)$$

A plot of  $1/T E|N(f)|^2$ , the average power density of the time limited noise, is shown in Fig. C-1 for several different  $BT$  products. Note that as  $BT \rightarrow \infty$  this function approaches the "rect," and for  $BT > 10$  a "rect" is a good approximation to the actual power density. This can be seen directly from Eq. (C6) by noting that compared to the "rect" function the squared "sinc" function is becoming very narrow, i. e., approaching an impulse as  $BT \rightarrow \infty$ . Since this "impulse" has area  $1/T$ , we can use the approximation

$$E|N(f)|^2 = \frac{\sigma^2 T}{2B} \text{rect}\left(\frac{f}{2B}\right) \quad (C7)$$

With these preliminaries taken care of, we can now turn to the verification of Eqs. (14) and (15).

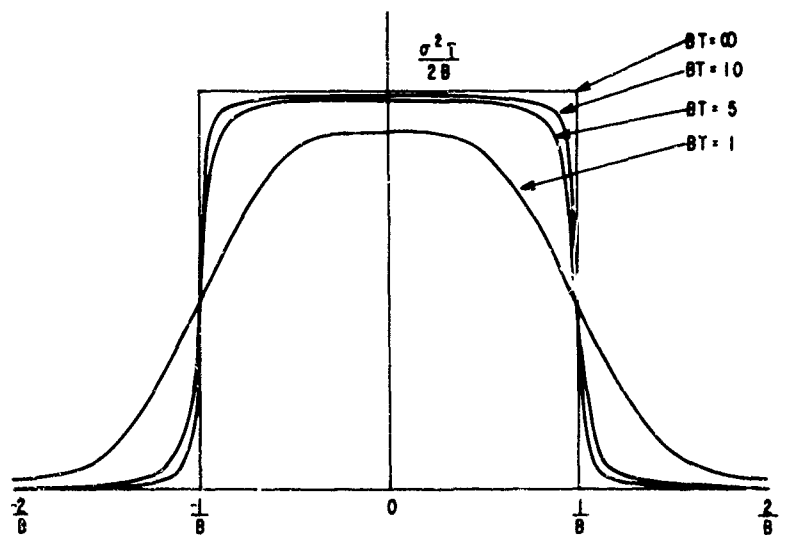


Fig. C-1 Average power density of band-limited white noise for various  $BT$  products.

From Eqs. (12) and (13) we note that  $V'(P_0)$  and  $V''(P_0)$  are needed.

$$V'(P_0) = \sum_k \frac{d}{dP} |Z\left(\frac{k}{P_0}\right)|^2 = \sum_k \frac{d}{dP} \left\{ |X\left(\frac{k}{P_0}\right)|^2 + |N\left(\frac{k}{P_0}\right)|^2 + 2\operatorname{Re}\left[X\left(\frac{k}{P_0}\right)N\left(\frac{k}{P_0}\right)] \right\} \quad (C8)$$

For large signal-to-noise ratios, the last term is much larger than the middle, and we will drop the middle term. Since  $P_0$  is the true period,  $|X(f)|^2$  will have a maximum at each frequency  $k/P_0$ , so the first term is zero. We are left with

$$V'(P_0) \cong \sum_k \frac{d}{dP} 2\operatorname{Re}[X(k/P_0)N(k/P_0)] \quad (C9)$$

Also, for large signal-to-noise ratios, we may neglect the contribution due to the noise, compared with that of the signal in calculating

$$V''(P_0) \cong \sum_k \frac{d^2}{dP^2} |X\left(\frac{k}{P_0}\right)|^2 \quad (C10)$$

We work on  $V'(P)$  first. Let

$$A(f) \cong 2\operatorname{Re}[X(f)N(f)] = X^*(f)N(f) + N^*(f)X(f) \quad (C11)$$

so

$$\begin{aligned} \frac{d}{df} A(f) &= A'(f) = X^*(f)N'(f) + X^{*'}(f)N(f) \\ &\quad + X(f)N^{*'}(f) + X'(f)N^*(f) \end{aligned} \quad (C12)$$

Now both the real and imaginary parts of  $X(f)$  are at a maximum at the frequencies  $k/P_0$ , so  $X'(k/P_0) = 0$  and

$$\begin{aligned} A'(k/P_0) &= X^*(k/P_0)N'(k/P_0) + X(k/P_0)N^{*'}(k/P_0) \\ V'(P_0) &\cong \sum_k \frac{d}{dP} A(k/P_0) = \sum_k \frac{-k}{P_0^2} A'(k/P_0) \end{aligned} \quad (C13)$$

$A'(f)$  is a random variable, linearly related to  $N'(f)$

$$N(f) = \int_{-\infty}^{\infty} \operatorname{rect}(t/T) \tilde{n}(t) e^{-j2\pi ft} dt$$

so

$$N'(f) = \int_{-\infty}^{\infty} \text{rect}(t/T) \tilde{n}(t) (-j2\pi t) e^{-j2\pi ft} dt$$

Thus

$$EN'(f) = \int_{-\infty}^{\infty} \text{rect}(t/T) E\tilde{n}(t) (-j2\pi t) e^{-j2\pi ft} dt = 0$$

$$\text{Hence } EV'(P_0) = 0 \quad (C14)$$

Since  $V''(P_0)$  as given by Eq. (C10) is not a random variable, Eq. (C14) shows that the expectation of the second term on the right of Eq. (13) is zero. This verifies Eq. (14).

Turning now to the variance of  $\hat{P}$ , we see from Eq. (13) that it depends upon

$$\text{Var } V'(P_0) = E[V'(P_0)]^2 = \sum_k \sum_l \frac{-k}{P_0^2} \frac{-l}{P_0^2} E \left[ A' \left( \frac{k}{P_0} \right) A' \left( \frac{l}{P_0} \right) \right] \quad (C15)$$

Now

$$\begin{aligned} E[A'(f)A'(g)] &= X_0^*(f)X_0^*(g)E[N'(f)N'(g)] + X_0(f)X_0(g)E[N'^*(f)N'^*(g)] \\ &\quad + X_0^*(f)X_0(g)E[N'(f)N'^*(g)] + X_0(f)X_0^*(g)E[N'^*(f)N'(g)] \end{aligned} \quad (C16)$$

We will evaluate one of these terms

$$\begin{aligned} E[N'(f)N'(g)] &= E \int_{-\infty}^{\infty} \text{rect}(t/T) (-j2\pi t) \tilde{n}(t) e^{-j2\pi ft} dt \cdot \\ &\quad \int_{-\infty}^{\infty} \text{rect}(s/T) (-j2\pi s) \tilde{n}(s) e^{-j2\pi gs} ds \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-j2\pi t)(-j2\pi s) \text{rect}(t/T) \text{rect}(s/T) R(t-s) e^{-j2\pi(ft+gs)} dt ds \\ &= \int_{-\infty}^{\infty} (-j2\pi t) \text{rect}(t/T) e^{-j2\pi ft} dt \cdot \int_{-\infty}^{\infty} (-j2\pi s) \text{rect}(s/T) e^{-j2\pi gs} R(t-s) ds \\ &= \int_{-\infty}^{\infty} (-j2\pi t) \text{rect}(t/T) e^{-j2\pi ft} dt \cdot \int_{-\infty}^{\infty} T^2 \text{sinc}' T(h+g) P(h) e^{j2\pi ht} dh \\ &= \int_{-\infty}^{\infty} T^2 \text{sinc}' T(h+g) P(h) dh \cdot \int_{-\infty}^{\infty} (-j2\pi t) \text{rect}(t/T) e^{-j2\pi(h-f)t} dt \\ &= \int_{-\infty}^{\infty} T^2 \text{sinc}' T(h+g) \cdot T^2 \text{sinc}' T(h-f) P(h) dh \end{aligned}$$

Substituting for  $P(h)$  from assumption (iv) in the Introduction,

$$E[N'(f)N'(g)] \cong \frac{\sigma^2}{2B} T^3 \text{sinc}^n T(g+f) \text{rect}(g/2B) \text{rect}(f/2B) \quad (C17)$$

because the sinc function behaves similar to a  $\delta$  function for large  $BT$ . Since the above is real,  $E[N^*(f)N^*(g)] = E[N'(f)N'(g)]$ . In a similar manner,

$$E[N'(f)N^{*'}(g)] = E[N^{*'}(f)N'(g)] \cong \frac{\sigma^2}{2B} T^3 \text{sinc}^n T(f-g) \text{rect}(g/2B) \text{rect}(f/2B) \quad (C18)$$

Thus

$$\begin{aligned} E[A'(f)A'(g)] &\cong \{[X_0^*(f)X_0^*(g) + X_0(f)X_0(g)] \text{sinc}^n T(f+g) \\ &\quad + [X_0(f)X_0^*(g) + X_0^*(f)X_0(g)] \text{sinc}^n T(f-g)\} \\ &\quad \frac{\sigma^2 T^3}{2B} \text{rect}(f/2B) \text{rect}(g/2B) \quad (C19) \end{aligned}$$

For  $f=g$ ,

$$\begin{aligned} E[A'(f)]^2 &= 2|X_0(f)|^2 \frac{\sigma^2 T^3}{2B} \text{sinc}^n(0) \text{rect}(f/2B) \\ &\quad + 2\text{Re}[X_0^2(f)] \frac{\sigma^2 T^3}{2B} \text{sinc}^n(2Tf) \text{rect}(f/2B) \quad (C20) \end{aligned}$$

For  $f = -g$  we get the same expression, so

$$\begin{aligned} \text{Var } V'(P) &\cong 2 \sum_{|k| < BP_0} \frac{k^2}{P_0^4} \left[ 2|X_0\left(\frac{k}{P_0}\right)|^2 \frac{\sigma^2 T^3}{2B} \frac{\pi^2}{3} + \right. \\ &\quad \left. 2\text{Re} \left[ X_0^2\left(\frac{k}{P_0}\right) \frac{\sigma^2 T^3}{2B} \right] \text{sinc}^n \left( 2T \frac{k}{P_0} \right) \right] \\ &\quad + \sum_{|k| < BP_0} \sum_{|\ell| < BP_0} \frac{k}{P_0} \frac{\ell}{P_0} E \left[ A'\left(\frac{k}{2B}\right) A'\left(\frac{\ell}{2B}\right) \right] \\ &\quad |k| \neq |\ell| \end{aligned}$$

where we have summed over both diagonals separately. We may neglect the last two sets of terms, in comparison with the first since they are down by a factor of at least  $P_0/T$ . Thus

$$\text{Var } V'(P) \cong \frac{2\pi^2}{3} \frac{T^3}{B} \frac{\sigma^2}{P_0^4} \sum_{|k| < BP_0} k^2 \left| X\left(\frac{k}{P_0}\right) \right|^2 \quad (C21)$$

The denominator of the second term on the right of Eq. (13) is

$$V''(P_0) \cong \frac{d^2}{dP^2} \sum_k \left| X_0\left(\frac{k}{P_0}\right) \right|^2 \Big|_{P_0}$$

which can be shown to be

$$V''(P_0) \cong \sum_k \frac{k^2}{P_0^4} \left| X_0\left(\frac{k}{P_0}\right) \right|^2 \cong \sum_k \frac{k^2}{P_0^4} \cdot \frac{2T^2\pi^2}{3} \left| X_0\left(\frac{k}{P_0}\right) \right|^2 \quad (C22)$$

Thus the variance of the estimator of the period  $P$  is

$$\begin{aligned} \text{Var } \hat{P} &= \left[ \frac{1}{\text{Var}''(P_0)} \right]^2 \text{Var } V'(P_0) \\ &= \frac{\frac{2\pi^2}{3} \frac{T^3}{B} \frac{\sigma^2}{P_0^4} \sum_{|k| < BP_0} k^2 \left| X_0\left(\frac{k}{P_0}\right) \right|^2}{\frac{4T^4\pi^4}{9P_0^8} \left[ \sum_{|l| < BP_0} k^2 \left| X_0\left(\frac{l}{P_0}\right) \right|^2 \right]^2} \\ &= \frac{3P_0^4}{\pi^2} \frac{1}{2BT} \frac{\sigma^2}{\sum_{|k| < BP_0} k^2 \left| X_0\left(\frac{k}{P_0}\right) \right|^2} \quad (C23) \end{aligned}$$

Now  $\left| X_0\left(\frac{k}{P_0}\right) \right|^2 = (T^2/P_0^2) |W_0(k/P_0)|^2$  and since

$$\frac{1}{P_0} \sum_{|k| < BP} (k^2/P_0^2) |W_0(k/P_0)|^2 \cong \int_{-B}^B f^2 |W_0(f)|^2 df \quad (C24)$$

Note that the integral is the second moment of the power spectrum of  $w(t)$  about  $f=0$ . We could appropriately define the bandwidth  $B_w$  of  $w(t)$  by the positive square root of

$$B_w^2 = \frac{1}{E_w} \int_{-B}^B f^2 |W(f)|^2 df \quad (C25)$$

where

$$E_w = \int_{-B}^B |W(f)|^2 df = \int_0^P w^2(t) dt \quad (C26)$$

Then the integral on the right of Eq. (C24) is  $B_w^2 E_w$ . Substituting this result back into Eq. (C23) gives Eq. (15).

## APPENDIX D

We shall verify Eqs. (19) and (20). We shall first verify Eq. (19) by finding the periodic waveform  $\hat{x}_P(t)$  which minimizes  $L_P$  given by Eq. (18). Let

$$X_P(t) = \sum_{m=0}^{(N-1)P} w_P(t-nP)$$

where  $w_P(t) = 0; t < 0, t \geq P$  .

Now let  $\eta(t)$  be a function like  $w_P(t)$

$$\text{i.e., } \eta(t) = 0; t < 0, t \geq P \quad .$$

Then let

$$w_P(t) = \hat{w}_P(t) + \epsilon \eta(t)$$

where  $\epsilon$  is a real variable. Since  $\hat{w}_P(t)$  minimizes  $L_P$ , it is necessary that

$$L_P(\epsilon) = \frac{1}{NP} \int_0^{NP} \left[ z(t) - \sum_{n=0}^N \hat{w}_P(t-nP) - \epsilon \eta(t-nP) \right]^2 dt$$

be stationary with respect to an incremental change in  $\epsilon$  when  $\epsilon = 0$ . Thus

$$\begin{aligned} 0 &= - \frac{1}{2} \frac{L_P(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} \\ &= \frac{1}{NP} \int_0^{NP} \left[ z(t) - \sum_{m=0}^N \hat{w}_P(t-mP) \right] \sum_{n=0}^N \eta(t-nP) dt \end{aligned}$$

Now if we multiply out the integrand and use the fact that  $\hat{w}_P(t-mP)$  and  $\eta(t-nP)$  are in disjoint  $P$  length intervals when  $m \neq n$ , the right-hand side becomes

$$\begin{aligned} &\frac{1}{NP} \int_0^{NP} \left\{ \sum_{n=0}^N z(t) \eta(t-nP) - \sum_{n=0}^N \hat{w}_P(t-nP) \eta(t-nP) \right\} dt \\ &= \frac{1}{NP} \sum_{n=0}^N \int_0^{NP} [z(t) - \hat{w}_P(t-nP)] \eta(t-nP) dt \quad . \end{aligned}$$

Now make the substitution of variables

$$s = t-nP \quad .$$



Then the integral becomes

$$\frac{1}{NP} \sum_{n=0}^N \int_{nP}^{NP-nP} [z(s+nP) - \hat{w}_P(s)] \eta(s) ds \quad .$$

Since  $\eta(s)$  is zero except in the interval  $0 \leq s < P$ , this expression can be written

$$\frac{1}{NP} \sum_{n=0}^{N-1} \int_0^P [z(s+nP) - \hat{w}_P(s)] \eta(s) ds \quad .$$

Taking the summation back inside the first integral, and making use of the gate function notation, the expression becomes

$$\frac{1}{NP} \int_0^P \left[ \sum_{n=0}^{N-1} z(t+nP) g_P(t) - N\hat{w}_P(t) \right] \eta(t) dt$$

clear that if this expression is to be zero for all possible  $\eta(t)$ , then it is required that

$$\hat{w}_P(t) = \frac{1}{N} \sum_{n=0}^{N-1} z(t+nP) g_P(t) \quad .$$

Or, stated in another form

$$\hat{w}_P(t) = \frac{1}{N} \sum_{n=0}^{N-1} z(t+nP) \quad ; 0 \leq t < P \quad (D1)$$

which is Eq. (19). This is indeed a minimum since

$$\frac{\partial^2 L_P(\epsilon)}{\partial \epsilon^2} \Big|_{\epsilon > 0} > 0 \quad \text{for } \eta(t) \neq 0$$

In order to verify Eq. (20) it is necessary to first repeat the above minimization process over the data intervals  $T$ , rather than  $NP$ . The result is

$$\hat{w}_P(t) = \begin{cases} \frac{1}{N+1} \sum_{n=0}^N z(t+nP) & ; 0 \leq t < \Delta \\ \frac{1}{N} \sum_{n=0}^{N-1} z(t+nP) & ; \Delta \leq t < P \end{cases} \quad (D2)$$

This equation shows that  $\hat{w}_P(t)$  differs from  $w_P(t)$  only in that the "left-over" portion of  $z(t)$  between  $NP$  and  $T$  has been averaged in also. The error between the two estimates is given by

$$\begin{aligned}
 & \hat{w}_P^1(t) - \hat{w}_P(t) \\
 &= \frac{1}{N+1} \sum_{n=0}^N z(t+nP) - \frac{1}{N} \sum_{n=0}^{N-1} z(t+nP) && ; 0 \leq t < (T-NP) \\
 &= \frac{1}{N+1} z(t+NP) + \left( \frac{1}{N+1} - \frac{1}{N} \right) \sum_{n=0}^N z(t+nP) && ; 0 \leq t < (T-NP) \\
 &= \frac{1}{N+1} \left[ z(t+NP) - \frac{1}{N} \sum_{n=0}^N z(t+nP) \right] && ; 0 \leq t < (T-NP) \\
 &= \frac{1}{N+1} [z(t+NP) - \hat{w}_P(t)] && ; 0 \leq t < (T-NP)
 \end{aligned}$$

and zero elsewhere. This is Eq. (20).

## APPENDIX E

Equation (22) is to be verified. First we note that

$$\begin{aligned}\int_0^{NP} \hat{x}_P(t) \hat{e}_P(t) dt &= \int_0^{NP} \hat{x}_P(t) [z(t) - \hat{x}_P(t)] dt \\ &= \int_0^{NP} \hat{x}_P(t) z(t) dt - \int_0^{NP} \hat{x}_P^2(t) dt\end{aligned}\quad (E1)$$

Second,  $\hat{x}_P(t)$  can be expanded as follows:

$$\begin{aligned}\hat{x}_P(t) &= \sum_{m=0}^{N-1} \hat{w}_P(t-mP) \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} z[t+(n-m)P] g_P(t-mP)\end{aligned}\quad (E2)$$

Now the two terms on the right of Eq. (E1) will be shown to cancel. Expanding the first of these with the aid of Eq. (E2) gives

$$\begin{aligned}\int_0^{NP} z(t) \hat{x}_P(t) dt &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^{NP} z(t) z[t+(n-m)P] g_P(t-mP) dt \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_{mP}^{mP+P} z(t) z[t+(n-m)P] dt\end{aligned}\quad (E3)$$

The second term on the right of Eq. (E2) gives

$$\begin{aligned}\int_0^{NP} \hat{x}_P^2(t) dt &= N \int_0^P \hat{w}_P^2(t) dt \\ &= N \int_0^P \left[ \frac{1}{N} \sum_{n=0}^{N-1} z(t+nP) g_P(t) \right]^2 dt \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^P z(t+mP) z(t+nP) dt\end{aligned}$$

Substituting  $s = t+mP$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_{mP}^{mP+P} z(s) z[s+(n-m)P] ds \quad (E4)$$

Since the right side of Eq. (E3) is identical to the right side of Eq. (E4), therefore the right side of Eq. (E1) is zero. Thus, Eq. (22) is verified.

## APPENDIX F

We wish to show that Eqs. (24) and (3) are equivalent definitions of  $V(P)$ . From Eq. (24),

$$V(P) = \int_0^{NP} \hat{x}_P^2(t) dt = \int_{-\infty}^{\infty} |\hat{X}_P(f)|^2 df$$

The second equality is from the Parseval theorem. Now by definition

$$\begin{aligned} \hat{x}_P(t) &= \sum_{n=0}^{N-1} \hat{w}_P(t-nP) \\ &= g_{NP}(t) \left[ \hat{w}_P(t) * \sum_n \delta(t-nP) \right] \\ &= g_{NP}(t) \left\{ \left[ \frac{1}{N} \left[ z(t) * \sum_m \delta(t+mP) \right] g_P(t) \right] * \sum_n \delta(t-nP) \right\}. \end{aligned}$$

Taking the Fourier transform of both sides gives

$$\begin{aligned} \hat{X}_P(f) &= NP \operatorname{sinc}(NPf) e^{-j\pi NPf} * \left( \frac{1}{N} \left[ Z(f) \sum_k \frac{1}{P} \delta\left(f - \frac{k}{P}\right) \right] \right. \\ &\quad \left. * P \operatorname{sinc}(Pf) e^{-j\pi Pf} \right\} \sum_{\ell} \frac{1}{P} \delta\left(f - \frac{\ell}{P}\right) \\ &= \operatorname{sinc}(NPf) e^{-j\pi NPf} * \left\{ \left[ \sum_k Z\left(\frac{k}{P}\right) \operatorname{sinc} P\left(f - \frac{k}{P}\right) e^{-j\pi\left(f - \frac{k}{P}\right)} \right] \sum_{\ell} \delta\left(f - \frac{\ell}{P}\right) \right\} \\ &= \operatorname{sinc}(Tf) e^{-j\pi NPf} * \left[ \sum_k \sum_{\ell} Z\left(\frac{k}{P}\right) \operatorname{sinc} P\left(\frac{\ell-k}{P}\right) e^{-j\pi P\left(\frac{\ell-k}{P}\right)} \right] \\ &= \sum_k Z\left(\frac{k}{P}\right) \operatorname{sinc} NP\left(f - \frac{k}{P}\right) e^{-j\pi NP\left(f - \frac{k}{P}\right)} \quad (F2) \end{aligned}$$

Then

$$\hat{X}_P^2(f) = \sum_k \sum_{\ell} Z\left(\frac{k}{P}\right) Z^*\left(\frac{\ell}{P}\right) \operatorname{sinc} NP\left(f - \frac{k}{P}\right) \operatorname{sinc} NP\left(f - \frac{\ell}{P}\right)$$

and

$$V(P) = \int |X_P^{\hat{}}(f)|^2 df = \frac{1}{NP} \sum_k |Z\left(\frac{k}{P}\right)|^2$$

which was to be shown.

## APPENDIX G

In this Appendix we shall verify Eq. (25). Once a  $\hat{P}$  has been chosen as the period estimate, the waveform estimate is

$$\hat{w}(t) = \frac{1}{N} \sum_{n=0}^{N-1} z(t+n\hat{P}) g_{\hat{P}}^*(t) \quad . \quad (G1)$$

Letting  $\hat{P} = P_0 + \hat{h}$ , we can expand  $z$  in a Taylor series about  $P_0$ .

$$\hat{w}(t) = \frac{1}{N} \sum_{n=0}^{N-1} [z(t+nP_0) + z'(t+nP_0)n\hat{h} + \dots] g_{\hat{P}}^*(t) \quad .$$

For large signal-to-noise ratio, we can assume that  $n\hat{h} \ll 1$ , in which case

$$\hat{w}(t) \approx \frac{1}{N} \sum_{n=0}^{N-1} [z(t+nP_0)] g_{\hat{P}}^*(t) \quad .$$

The expectation of  $\hat{w}(t)$  is then

$$E\hat{w}(t) = \frac{1}{N} \sum_{n=0}^{N-1} x(t+nP_0) E g_{\hat{P}}^*(t) + E[n(t+nP_0) g_{\hat{P}}^*(t)] \quad . \quad (G2)$$

Now we shall evaluate each of the expectations on the right separately.

$$E g_{\hat{P}}^*(t) = \int_{-\infty}^{\infty} g_{\hat{P}}^*(t) p(\hat{P}) d\hat{P} \quad ,$$

where  $p(\hat{P})$  is the density function of the period estimate. Unfortunately, it is necessary to have a probability distribution for  $\hat{P}$  to evaluate  $E g_{\hat{P}}^*(t)$ . Therefore, we shall proceed with the assumption that the noise is Gaussian. Let

$$\hat{P} = P_0 + \hat{h} \quad .$$

From Eq. (13) we have that

$$\hat{h} \approx - \frac{V'(P_0)}{V''(P_0)} \quad .$$

By our approximations  $V''(P_0)$  is given by Eq. (C9) and is not a function of the noise. However, Eqs. (C8) and (C7) show  $V'(P_0)$  to be a linear function of the noise. Thus  $\hat{h}$  is (to a first approximation) a Gaussian random variable, with zero mean and variance  $\sigma_P^2$  given by Eq. (15). Then

$$p(\hat{P}) = \varphi_{P_0, \sigma_P}(\hat{P})$$

where  $\varphi$  is the Gaussian density function with mean  $P_0$  and variance  $\sigma_P^2$ . Then

$$E\hat{g}_P(t) = \int_{-\infty}^{\infty} \hat{g}_P(t) \varphi_{P_0, \sigma_P}(\hat{P}) d\hat{P}.$$

For any  $t < 0$ ,  $\hat{g}_P(t) = 0$ .

For any  $t > 0$ ,

$$\hat{g}_P(t) = \begin{cases} 1 & \text{for } \hat{P} \geq t \\ 0 & \text{for } \hat{P} < t \end{cases}.$$

Hence,

$$E\hat{g}_P(t) = \begin{cases} \int_t^{\infty} \varphi_{P_0, \sigma_P}(\hat{P}) d\hat{P} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

Substituting  $x = (\hat{P} - P_0)/\sigma_P$  gives

$$E\hat{g}_P(t) = \begin{cases} \int_{\frac{t-P_0}{\sigma_P}}^{\infty} \varphi_{0,1}(x) dx & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$= \begin{cases} 1 - \Phi\left(\frac{t-P_0}{\sigma_P}\right) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where  $\Phi$  is the Gaussian distribution function. This allows us to evaluate the first term of Eq. (G2). The second term of Eq. (G2) is

$$E[n(t+nP_0) \hat{g}_P(t)]$$

which is essentially the average value of the noise over an interval  $\hat{P}$ . Since the expected value of the noise is zero, it is reasonable to expect this term to be nearly zero. Hence

$$E\hat{W}(t) \approx \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} x(t+nP_0) \left[1 - \Phi\left(\frac{t-P_0}{\sigma_P}\right)\right] & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

## REFERENCES AND FOOTNOTES

1. H. D. Brunk, An Introduction to Mathematical Statistics, Ginn and Co., Boston (1960), Chap. 8.
2. M. Fisz, Probability Theory and Mathematical Statistics, John Wiley and Sons, Inc., New York (1963), Chap. 13.
3. All summations are from  $-\infty$  to  $\infty$  unless otherwise indicated.
4. This method was suggested by Helstrom's treatment of a similar case. See C. W. Helstrom, Statistical Theory of Signal Detection, Pergamon Press, New York (1960), p. 231.
5. It can be shown that the change in the first term with  $P$  becomes important when  $z^2(NP)$  exceeds the mean square value of  $z(t)$  by a factor of  $N^2$  or more.

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